Assume $a+b+c \le -3$. The given condition $a \ge 1$ implies $b+c \le -4$, whence $-3 < c \le b < -1$. Using these inequalities, as well as $b+c<0$, we have

$$ab+bc+ca = a(b+c)+bc \le b+c+bc = (b+1)(c+1)-1 < (-2)(-2)-1 = 3.$$

Thus $a+b+c \le -3$ implies $ab+bc+ca < 3$, contradicting $ab+bc+ca=3$. It follows that $a+b+c \ge 3$, with equality if and only if $a=b=c=1$.

4000. *Proposed by Marcel Chiriţă.*

Let $x_1, x_2, \ldots x_n$ with $x_1 > x_2 > \ldots > x_n > 0$, $x_1 x_2 \ldots x_n = 1$ and $n \ge 3$. Show that

$$\frac{x_1^2+x_2^2}{x_1-x_2}\cdot\frac{x_2^2+x_3^2}{x_2-x_3}\cdots\frac{x_{n-1}^2+x_n^2}{x_{n-1}-x_n}\cdot\frac{x_1^2+x_n^2}{x_1-x_n} > 2^{3/n}.$$

While the inequality holds as written, the right hand side was supposed to be the much stronger $2^{3n/2}$; we apologize for the error. As a consequence however, the solutions we received (11 in total) varied greatly in their approach. The strength of the proven results also varied, with the right hand side being variously replaced by $2^{n/3}$, 2^{n-1}, 2^n or $(2+2\sqrt{2})^{n-1}$, as well as the intended $2^{3n/2}$.

We present two of the solutions.

Solution 1, by Arkady Alt, modified by the editor. This solution had the best bound.

For $1 \le k \le n-1$, let $t_k := \frac{x_k}{x_{k+1}}$. Note that $t_k > 1$, and

$$\frac{x_k^2+x_{k+1}^2}{x_k-x_{k+1}} = \frac{x_{k+1}^2\left(\frac{x_k^2}{x_{k+1}^2}+1\right)}{x_{k+1}\left(\frac{x_k}{x_{k+1}}-1\right)} = x_{k+1}\cdot\frac{\left(\frac{x_k}{x_{k+1}}\right)^2+1}{\frac{x_k}{x_{k+1}}-1} = x_{k+1}\cdot\frac{t_k^2+1}{t_k-1}.$$

Let $a := \frac{x_n}{x_1}$ and note $0<a<1$; then

$$\frac{x_1^2+x_n^2}{x_1-x_n} = \frac{x_1^2(1+a^2)}{x_1(1-a)} = x_1\cdot\frac{1+a^2}{1-a}.$$

Since $x_1 x_2 \ldots x_n = 1$, it follows that

$$\prod_{k=1}^{n-1}\frac{x_k^2+x_{k+1}^2}{x_k-x_{k+1}}\cdot\frac{x_1^2+x_n^2}{x_1-x_n} = \prod_{k=1}^{n-1}\frac{t_k^2+1}{t_k-1}\cdot\frac{a^2+1}{1-a}. \tag{1}$$

For any $t>1$ we claim that $\frac{t^2+1}{t-1} \ge 2\sqrt{2}+2$.
To see this, note that

$$\begin{aligned} t^2+1-(2\sqrt{2}+2)(t-1) &= t^2+1-2\sqrt{2}t-2t+2\sqrt{2}+2 \\ &= (t-\sqrt{2}-1)^2 \ge 0, \end{aligned}$$

whence, dividing by $t-1$, we can obtain the claimed inequality.

On the other hand, for $0<a<1$, it is easy to check that $\frac{a^2+1}{1-a} > 1$.

In (1), we had $t_k > 1$ and $a < 1$, so using the last two observations, we get

$$\prod_{k=1}^{n-1} \frac{x_k^2 + x_{k+1}^2}{x_k - x_{k+1}} \cdot \frac{x_1^2 + x_n^2}{x_1 - x_n} > (2\sqrt{2} + 2)^{n-1}.$$

Solution 2, by Joel Schlosberg, slightly modified by the editor. This solution addressed the question of when the minimum is achieved.

Suppose $a, b \in \mathbb{R}$ with $0 < a < b$. Consider the function $g(x) = \frac{(b^2+x^2)(x^2+a^2)}{x(b-x)(x-a)}$. We claim that on the interval (a, b), $g(x)$ has a unique minimum at $x = \sqrt{ab}$.

Multiplying,

$$g(x) = \frac{b^2x^2 + b^2a^2 + x^4 + x^2a^2}{x(bx + xa - (ba + x^2))}.$$

For $x > 0$ we can use the arithmetic mean-geometric mean inequality. In the numerator, we have

$$b^2a^2 + x^4 \geq 2\sqrt{b^2a^2 \cdot x^4} = 2bax^2$$

(with equality if and only if $x = \sqrt{ab}$), and in the denominator

$$ba + x^2 \geq 2\sqrt{bax^2} = 2x\sqrt{ba}$$

(once again, with equality if and only if $x = \sqrt{ab}$). Moreover, for $x \in (a, b)$ we have that the denominator

$$x(b - x)(x - a) = x(bx + xa - (ba + x^2)) > 0$$

and hence

$$g(x) \geq \frac{b^2x^2 + x^2a^2 + 2bax^2}{x(bx + xa - 2x\sqrt{ba})} = \frac{b^2 + a^2 + 2ba}{b + a - 2\sqrt{ba}}.$$

Note that the right-hand side does not depend on x, and equality holds if and only if $x = \sqrt{ab}$. This concludes the proof that, for $x \in (a, b)$, $g(x)$ has a unique minimum at $x = \sqrt{ab}$. Note that g is differentiable on (a, b), and so $x = \sqrt{ab}$ must also be the unique critical point of g on (a, b).

Now suppose $z_1, z_n \in \mathbb{R}$ are fixed, with $z_1 > z_n > 0$, and define $f : \mathbb{R}^{n-2} \to \mathbb{R}$ by

$$f : (z_2, \ldots, z_{n-1}) \to \frac{1}{z_1 \cdot \ldots \cdot z_n} \cdot \frac{z_1^2 + z_2^2}{z_1 - z_2} \cdot \ldots \cdot \frac{z_{n-1}^2 + z_n^2}{z_{n-1} - z_n}.$$

Consider the domain

$$D := \{(t_2, \ldots, t_{n-1}) \in \mathbb{R}^{n-2} : z_1 > t_2 > \ldots > t_{n-1} > z_n\}.$$

As $(z_2, \ldots, z_{n-1})$ approaches the boundary of D, some $z_{i-1} - z_i \to 0$, and so $f \to +\infty$. Since f is real-valued and continuous on the open domain D, an absolute minimum value of f must be attained at a critical point. The partial derivatives